THREEFOLD EXTREMAL CONTRACTIONS OF TYPES (IC) AND (IIB)

SHIGEFUMI MORI AND YURI PROKHOROV

Abstract. Let (X, C) be a germ of a threefold X with terminal singularities along an irreducible reduced complete curve C with a contraction $f : (X, C) \to (Z, o)$ such that $C = f^{-1}(o)_{\mathrm{red}}$ and $-K_X$ is ample. Assume that (X, C) contains a point of type (IC) or (IIB). We complete the classification of such germs in terms of a general member $H \in |\mathscr{O}_X|$ containing C.

1. Introduction

1.1. Let (X, C) be a germ of a threefold with terminal singularities along an reduced complete curve. We say that (X, C) is an *extremal curve germ* if there is a contraction $f : (X, C) \to (Z, o)$ such that $C = f^{-1}(o)_{\mathrm{red}}$ and $-K_X$ is f-ample.

If furthermore f is birational, then (X, C) is said to be an *extremal neighborhood* [Mor88]. In this case f is called *flipping* if its exceptional locus coincides with C (and then (X, C) is called *isolated*). Otherwise the exceptional locus of f is two-dimensional and f is called *divisorial*. If f is not birational, then $\dim Z = 2$ and (X, C) is said to be a $\mathbb{Q}$-*conic bundle germ* [MP08].

1.2. In this paper we consider only extremal curve germs with irreducible central fiber C. For each singular point P of X with $P \in C$, consider the germ $(P \in C' \subset X)$. All such germs are classified into types IA, IC, IIA, IIB, IA$^\vee$, II$^\vee$, ID$^\vee$, IE$^\vee$, and III, whose definitions we refer the reader to [KM92] and [MP08].

In this paper we complete the classification of extremal curve germs with irreducible central fiber containing points of type IC or IIB. As in [KM92] and [MP11] the classification is done in terms of a general hyperplane section, that is, a general divisor H of $|\mathscr{O}_X|_C$, the linear subsystem of $|\mathscr{O}_X|$ consisting of sections containing C.

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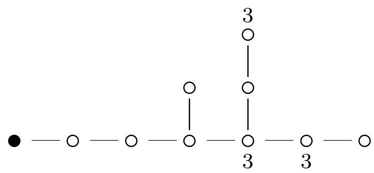
For a normal surface S and a curve $V \subset S$, we use the usual notation of graphs $\Delta(S, V)$ of the minimal resolution of S near V: each $\diamond$ corresponds to an irreducible component of V and each $\circ$ corresponds to an exceptional divisor on the minimal resolution of S, and we may use $\bullet$ instead of $\diamond$ if we want to emphasize that it is a complete (-1)-curve. A number attached to a vertex denotes the minus self-intersection number. For short, we may omit 2 if the self-intersection is -2.

Recall that if an extremal curve germ $(X, C \simeq \mathbb{P}^1)$ contains a point of type IC, then (X, C) is not divisorial [KM92, Cor. 8.3.3]. For the remaining $\mathbb{Q}$-conic bundle case we prove the following.

1.3. Theorem. *Let (X, C) is a $\mathbb{Q}$-conic bundle germ of type* (IC) *with irreducible C and let $f : (X, C) \to (Z, o)$ be the corresponding contraction. Let $P \in X$ be (a unique) singular point. Then we have:*

1.3.1. *The point $P \in X$ is of index* 5*. Moreover, the general member $H \in |\mathscr{O}_X|_C$ is normal, smooth outside of P, has only rational singularities, and the following is the only possibility for the dual graph of* (H, C):

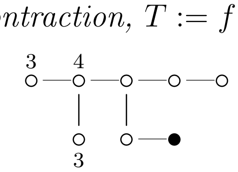


If an extremal curve germ $(X, C \simeq \mathbb{P}^1)$ contains a point of type (IIB), then it cannot be flipping [KM92, Theorem 4.5]. Remaining cases of divisorial contractions and $\mathbb{Q}$-conic bundles are covered by the following theorem.

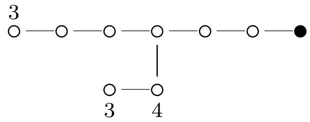
1.4. Theorem. *Let (X, C) is an extremal curve germ of type* (IIB) *with irreducible C and let $f : (X, C) \to (Z, o)$ be the corresponding contraction. Let $P \in X$ be (a unique) singular point. Then the general member $H \in |\mathscr{O}_X|_C$ is normal, smooth outside of P, and has only rational singularities. Moreover, the following are the only possibilities for the dual graph of (H, C).*

(X, P) **is a simple** cAx/4 **point (see 3.1.1):**

1.4.1. *f is a divisorial contraction, $T := f(H)$ is Du Val of type* A_2,

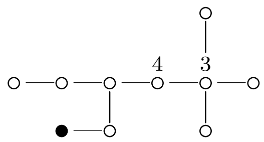


1.4.2. *f is divisorial contraction, $T := f(H)$ is smooth,*

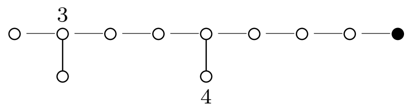


(X, P) **is a double** cAx/4 **point:**

1.4.3. *f is divisorial contraction,* $T := f(H)$ *is Du Val of type* D_4,



1.4.4. *f is a* $\mathbb{Q}$*-conic bundle*,



2. Case (IC)

In this section we prove Theorem 1.3. The techniques of [KM92, ch. 8] will be used freely, sometimes without additional explanations.

2.1. Setup. Let (X, P) be the germ of a three-dimensional terminal singularity and let $C \subset (X, C)$ be a smooth curve. Recall that the triple (X, C, P) is said to be of type (IC) if there are analytic isomorphisms

$$(X, P) \simeq \mathbb{C}^3_{y_1, y_2, y_4}/\boldsymbol{\mu}_m(2, m-2, 1), \quad C^\sharp \simeq \{y_1^{m-2} - y_2^2 = y_4 = 0\},$$

where m is odd and $m \geq 5$.

2.1.1. Let (X, C) be a $\mathbb{Q}$-conic bundle germ and let $f\colon (X, C) \to (Z, o)$ be the corresponding contraction. In this section we assume that C is irreducible and has a point P of type (IC). Recall that (X, C) is locally primitive at P [Mor88, 4.2]. Moreover, P is the only singular point on C [MP08, Theorem 8.6, Lemma 7.1.2]. Thus the group $\mathrm{Cl}(Z, o)$ has no torsion. Therefore, the base point (Z, o) is smooth.

2.2. We have an ℓ-splitting

$$\operatorname{gr}^1_C \mathscr{O} = (4P^\sharp) \mathbin{\tilde{\oplus}} (-1 + (m-1)P^\sharp) \tag{2.2.1}$$

by [MP09, §3], [KM92, 2.10.2], and hence the unique $(4P^\sharp)$ in $\operatorname{gr}^1_C \mathscr{O}$. Since y_4 and $y_1^{m-2} - y_2^2$ form an ℓ-free ℓ-basis of $\operatorname{gr}^1_C \mathscr{O}$ at P, $(4P^\sharp)$ has an ℓ-free ℓ-basis of the form

$$u = \lambda_1 y_1^{(m-5)/2} y_4 + \mu_1 (y_1^{m-2} - y_2^2) \tag{2.2.2}$$

for some λ_1 and $\mu_1 \in \mathscr{O}_{C,P}$. It is easy to see that whether $\lambda_1(P) \neq 0$ does not depend on the choice of coordinates.

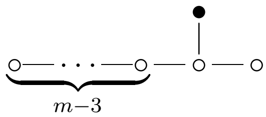
2.2.3. Remark. We have

$$\mathscr{O}_C = \mathscr{O}_C(-H) \hookrightarrow \operatorname{gr}^1_C \mathscr{O} = \mathscr{O} \oplus \mathscr{O}(-1)$$

If $m \geq 7$, this implies that the term $y_1^2(y_1^{m-2} - y_2^2)$ appears in the equation of H. If $m = 5$, then either $y_1^2(y_1^3 - y_2^2)$ or $y_1^2 y_4$ appears in the equation of H.

2.3. According to [MP09, §3] (cf. [KM92, 2.10]) a general member $F \in |-K_X|$ contains C, has only Du Val singularities, and $\Delta(F, C)$ is the following graph of (-2)-curves

(2.3.1)



where $\bullet$ corresponds to C. We can choose coordinates y_1, y_2, y_4 in a neighborhood of P so that $F = \{y_4 = 0\}/\boldsymbol{\mu}_m$. In particular, the ℓ-splitting (2.2.1) has the form

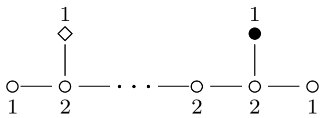
$$\operatorname{gr}^1_C \mathscr{O} = (4P^\sharp) \mathbin{\tilde{\oplus}} \mathscr{O}_C(-F). \tag{2.3.2}$$

2.4. Lemma. *A general member $H \in |\mathscr{O}_X|_C$ is normal, has only rational singularities, and smooth outside of P.*

Proof. Similar to 3.4.3. Let $T := f(H)$ and let $\Gamma := H \cap F$. As in 3.3.2 consider the Stein factorization

$$f_F : (F, C) \xrightarrow{f_1} (F_Z, o_Z) \xrightarrow{f_2} (Z, o). \tag{2.4.1}$$

Put $\Gamma_Z := f_1(\Gamma)$. We may assume that, in some coordinate system, the germ (F_Z, o_Z) is given by $z^2 + xy^2 + x^{m-1} = 0$. Then by [Cat87] up to coordinate change the double cover $(F_Z, o_Z) \longrightarrow (Z, o)$ is just the projection to the (x, y)-plane. Hence we may assume that Γ_Z is given by $x = y$. By 2.3 we see that the graph $\Delta(F, \Gamma)$ has the form



Therefore, Γ is reduced and so H is smooth outside of P. The restriction $f_H : H \to T$ is a rational curve fibration. Hence H has only rational singularities. $\square$

2.5. Let J be the C-laminal ideal such that $I_C \supset J \supset \mathrm{F}^2_C \mathscr{O}$ and $J/\mathrm{F}^2_C \mathscr{O} = (4P^\sharp)$ in (2.3.2). Since J is locally a nested c.i. on $C \setminus \{P\}$ and (y_4, u) is a (1,2)-monomializing ℓ-basis of $I_C \supset J$ at P with u as in (2.2.2). We have an ℓ-exact sequence

$$0 \to \mathscr{O}_C(-2F) \longrightarrow \operatorname{gr}^0_C J \longrightarrow (4P^\sharp) \to 0 \tag{2.5.1}$$

and an ℓ-isomorphism $\mathscr{O}_C(-2F) \simeq (-1 + (m-2)P^\sharp)$. Thus we have $\operatorname{gr}^0_C J \simeq \mathscr{O} \oplus \mathscr{O}(-1)$ as $\mathscr{O}_C$-modules. The unique $\mathscr{O}$ in $\operatorname{gr}^0_C J$ is generated near P by

$$y_1^2 u + \alpha y_2 y_4^2 \mod \mathrm{F}^3(\mathscr{O}, J) \tag{2.5.2}$$

for some $\alpha \in \mathscr{O}_{C,P}$.

Proofs of the following two lemmas given in [KM92] work in our situation without any changes.

2.6. Lemma ([KM92, Lemma 8.5.3]).

$$\mathrm{F}^3(\mathscr{O},J)^\sharp \subset \left((y_1^{m-2}-y_2^2)^2,\, (y_1^{m-2}-y_2^2)y_4,\, \lambda_1 y_1^{(m-5)/2}y_4^2,\, y_4^3\right).$$

2.7. Lemma ([KM92, Lemma 8.6]). *The ℓ-exact sequence* (2.5.1) *is ℓ-split if and only if $\alpha(P)=0$.*

2.8. Proposition. *If $m\ge 7$, then $\alpha(P)\ne 0$.*

Proof. Assume that $\alpha(P)=0$, that is, (2.5.1) is ℓ-split. Then $\mathrm{gr}^0_C J$ contains a unique $(4P^\sharp)$. Let $\mathscr{K}$ be the C-laminal ideal such that $J\supset\mathscr{K}\supset\mathrm{F}^1(\mathscr{O},J)$ and $\mathscr{K}/\mathrm{F}^1(\mathscr{O},J)=(4P^\sharp)$. By [Mor88, 8.14], $\mathscr{K}$ is locally a nested c.i. on $C\setminus\{P\}$ and $(1,3)$-monomializable at P, and we have ℓ-isomorphisms

$$\mathrm{gr}^i_C(\mathscr{O},\mathscr{K})\simeq(-1+(m-i)P^\sharp),\qquad i=1,\,2 \tag{2.8.1}$$

and an ℓ-exact sequence

$$0\to(-1+(m-3)P^\sharp)\longrightarrow\mathrm{gr}^3_C(\mathscr{O},\mathscr{K})\longrightarrow(4P^\sharp)\to 0. \tag{2.8.2}$$

By (2.8.1)$\tilde{\otimes}\omega_X$, we see $\mathrm{gr}^i_C(\omega_X,\mathscr{K})\simeq(-1+(m-i-1)P^\sharp)$ and so $H^j(\mathrm{gr}^i_C(\omega_X,\mathscr{K}))=0$ for $i=1,\,2$, $j=0,\,1$ because

$$m-2,\ m-3\in 2\mathbb{Z}_+ + (m-2)\mathbb{Z}_+.$$

Now using (2.8.2) $\tilde{\otimes}\omega_X$, we obtain

$$0\to(-2+(2m-4)P^\sharp)\longrightarrow\mathrm{gr}^3_C(\omega_X,\mathscr{K})\longrightarrow(-1+(m+3)P^\sharp)\to 0.$$

We note $(-1+(m+3)P^\sharp)\simeq\mathscr{O}(-1)$ as $\mathscr{O}_C$-modules because $3\notin 2\mathbb{Z}_+ + (m-2)\mathbb{Z}_+$ for $m\ge 7$. We similarly note that $(-2+(2m-4)P^\sharp)\simeq\mathscr{O}(-2)$ because $m-4\notin 2\mathbb{Z}_+ + (m-2)\mathbb{Z}_+$. Hence, $H^1(\mathrm{gr}^3_C(\omega_X,\mathscr{K}))\ne 0$. Note that $\omega_X/\mathrm{F}^1(\omega_X,\mathscr{K})=\mathrm{gr}^0_C\,\omega\simeq\mathscr{O}(-1)$. Using the standard exact sequences

$$0\to\mathrm{gr}^i_C(\omega_X,\mathscr{K})\longrightarrow\omega_X/\mathrm{F}^{i+1}(\omega_X,\mathscr{K})\longrightarrow\omega_X/\mathrm{F}^i(\omega_X,\mathscr{K})\to 0,$$

we obtain $H^1(\omega_X/\mathrm{F}^4(\omega_X,\mathscr{K}))\ne 0$. By [MP08, 4.4] we have

$$-K_X\cdot C=5/m\ge -K_X\cdot f^{-1}(o)=2,$$

a contradiction. □

2.9. Proposition.

(i) *$\mathscr{O}_F(-C)$ is an ℓ-invertible $\mathscr{O}_F$-module with an ℓ-free ℓ-basis $y_1^{m-2}-y_2^2$ at P and an ℓ-isomorphism.*

$$\mathscr{O}_C\,\tilde{\otimes}\,\mathscr{O}_F(-C)\simeq(4P^\sharp).$$

(ii) *$H^0(\mathscr{O}_F(-\nu C))\twoheadrightarrow H^0(\mathscr{O}_C\,\tilde{\otimes}\,\mathscr{O}_F(-\nu C))$ for all $\nu\ge 0$.*

(iii) *There are sections $s_1,\,s_2\in H^0(I_C)$ such that*

$$s_1\equiv(\mathrm{unit})\cdot(y_1+\xi_1 y_2^{m-1})^2(y_1^{m-2}-y_2^2)\quad\mathrm{mod}\ y_4\quad\text{near } P,$$

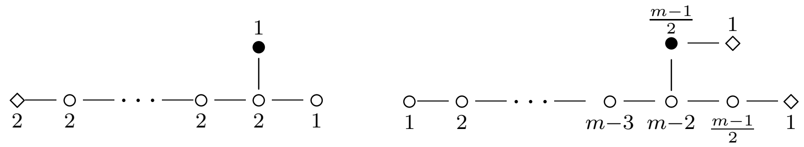
$$s_2\equiv(\mathrm{unit})\cdot(y_2+\xi_2 y_1^{m-1})(y_1^{m-2}-y_2^2)^{(m-1)/2}\quad\mathrm{mod}\ y_4\quad\text{near } P,$$

where $\xi_1,\,\xi_2\in\mathscr{O}_{X^\sharp}$ are invariants.

(iv) $H^0(I_C) \twoheadrightarrow H^0(\operatorname{gr}^0_C J) = H^0(I_C/\operatorname{F}^3(\mathscr{O}, J)) \simeq \mathbb{C}$.

Proof. (i) follows from the construction of F. Hence, $H^1(\mathscr{O}_C \tilde{\otimes} \mathscr{O}_F(-\nu C)) = 0$ for all $\nu \geq 0$, and $H^1(\mathscr{O}_F(-\nu C)) = 0$ since C is a fiber of proper f. Thus we have (ii).

To prove (iii) consider the Stein factorization (2.4.1) and as in the proof of Lemma 2.4 we take an embedding $(F_Z, o_Z) \subset \mathbb{C}^3_{x,y,z}$ so that (F_Z, o_Z) is given by the equation $z^2 + xy^2 + x^{m-1}$ and the map $f_2 : (F_Z, o_Z) \to (Z, o)$ is just the projection to the (x, y)-plane. Take $s_1 = f^*x$ and $s_2 = f^*y$. The weighted blowup of (F_Z, o_Z) with weights $(2, m-2, m-1)$ extracts the central vertex of the $\mathrm{D_m}$-diagram (2.3.1). The multiplicity of the corresponding exceptional curve in f_2^*x and f_2^*y is equal to 2 and $m-2$, respectively. Using this one can easily show that multiplicities of all exceptional curves in f_2^*x and f_2^*y, respectively, are given by the following diagrams



where the vertex $\bullet$, as usual, corresponds to C and vertices $\diamond$ correspond to components of the proper transforms of $\{f_2^*x = 0\}$ and $\{f_2^*y = 0\}$. The multiplicity of C is exactly the exponent of $y_1^{m-2} - y_2^2$ in $s_i \mod y_4$. Therefore,

$$s_1 \equiv \gamma_1(y_1^{m-2} - y_2^2) \qquad s_2 \equiv \gamma_2(y_1^{m-2} - y_2^2)^{(m-1)/2} \mod y_4,$$

where $\gamma_i \in \mathscr{O}_{X^\sharp}$ are semi-invariants. Using the above diagrams, we see $(\{\gamma_1 = 0\} \cdot C)_F = -4/m$ and $(\{\gamma_2 = 0\} \cdot C)_F = (m-2)/m$ because $(C^2)_F = 4/m$ by (i). Since $y_1 y_2$ is of weight 0, we have

$$\gamma_1 = (\text{unit}) \cdot (y_1 + y_2^{m-1}\xi_1)^2 \mod y_4$$

for some $\xi_1 \in \mathscr{O}_X$. Indeed, since $\gamma_1 = 0$ defines a double curve on F, one has $\gamma_1 = (\text{unit}) \cdot \delta^2 \mod y_4$ for some $\delta \in \mathscr{O}_{X^\sharp}$ with weight $\equiv 2$ such that $\delta|_C = y_1|_C$.

Similarly, we have $\gamma_2|_C = y_2|_C$. Hence,

$$\gamma_2 = (\text{unit}) \cdot (y_2 + y_1^{m-1}\xi_2) \mod y_4.$$

Finally, (iv) follows from (iii) because $H^0(\operatorname{gr}^0_C J) \simeq \mathbb{C}$. □

2.10. By Proposition 2.8 there are four cases to treat.

2.10.1. Case $m \geq 7$, $\alpha(P) \neq 0$.

2.10.2. Case $m = 5$, $\lambda_1(P) \neq 0$.

2.10.3. Case $m = 5$, $\lambda_1(P) = 0$, $\alpha(P) \neq 0$.

2.10.4. Case $m = 5$, $\lambda_1(P) = 0$, $\alpha(P) = 0$.

We shall show that cases 2.10.1, 2.10.2, 2.10.3 do not occur and 2.10.4 implies 1.3.1.

2.11. Proof of 1.3**; cases** 2.10.1 **and** 2.10.3**.** By (2.5.2) and Proposition 2.9, a general section $s \in H^0(I_C)$ satisfies

$$s \equiv (\text{unit}) \cdot \left(y_1^2 u + \alpha y_2 y_4^2\right) \mod \mathrm{F}^3(\mathscr{O}, J) \quad \text{at } P,$$

where $\alpha(P) \neq 0$ by assumption. Let us take s_2 given in (iii) of Proposition 2.9. We claim that s_2 belongs to $H^0(\mathrm{F}^3(\mathscr{O}, J))$. Indeed, it is obvious that $s \notin \mathbb{C}\cdot s_2 + \mathrm{F}^3(\mathscr{O}, J)$ near P. Hence by $H^0(I_C/\mathrm{F}^3(\mathscr{O}, J)) = \mathbb{C}\cdot s$, we have $s_2 \in H^0(\mathrm{F}^3(\mathscr{O}, J))$ as claimed. By Lemma 2.6, we see that the coefficient of $y_2 y_4^2$ (resp. y_2^m) in the Taylor expansion of s_2 at $P^\sharp$ is 0 (resp. non-zero) because $m \geq 7$ or $\lambda_1(P) = 0$. We now analyze the set $H = \{s = 0\}$. By Bertini's theorem, H is smooth outside of C. Since $\mathscr{O} \cdot s$ is the unique $\mathscr{O}$ in $\mathrm{gr}_C^1 \mathscr{O} \simeq \mathscr{O} \oplus \mathscr{O}(-1)$, H is smooth on $C \setminus \{P\}$. To study (H, P), we can apply [KM92, 10.7]. Indeed, if $\lambda_1(P) = 0$, then $\mu_1(P) \neq 0$ by the construction 2.2. Thus [KM92, 10.7.1] holds by Lemma 2.6. Replacing s with a general linear combination of s and s_2 we see that [KM92, 10.7.2] is satisfied. Since $m \geq 7$ or $\lambda_1(P) = 0$, we can now apply [KM92, 10.7]. One can see that the contraction $f_H : H \to T$ must be birational in this case, which is a contradiction.

2.12. Proof of 1.3**; case** 2.10.2**.** The argument is the same as 2.11 except that we need to check the conditions of [KM92, 10.7]. Note that (2.2.2) has the form $u = \lambda_1 y_4 + \mu_1(y_1^3 - y_2^2)$. Since $\lambda_1(P) \neq 0$, by a coordinate change we can make $\mu_1(P) \neq 0$. Let $D := \{y_1 = 0\}/\boldsymbol{\mu}_m \in |-2K_X|$ and let

$$\phi_D := \frac{u - \lambda_1(P) y_4}{\mathrm{d}\, y_1 \wedge \mathrm{d}\, y_2 \wedge \mathrm{d}\, y_4} = \frac{(\lambda_1 - \lambda_1(P)) y_4 + \mu_1(y_1^3 - y_2^2)}{\mathrm{d}\, y_1 \wedge \mathrm{d}\, y_2 \wedge \mathrm{d}\, y_4} \in \mathscr{O}_D(-K_X).$$

Arguments in [MP09, 3.1] show that there exists a section $\phi \in H^0(\mathscr{O}(-K_X))$ sent to ϕ_D modulo ω_Z. Thus the image of ϕ under the homomorphism

$$I_C \mathbin{\tilde{\otimes}} \mathscr{O}_X(-K_X) \twoheadrightarrow \mathrm{gr}_C^1 \mathscr{O}_X(-K_X) = (1) \mathbin{\tilde{\oplus}} (0) \twoheadrightarrow (0)$$

is non-zero because $\lambda_1(P) \neq 0$. Hence $F' = \{\phi = 0\} \in |-K_X|$ is smooth outside of P and we may choose ϕ so that F' is furthermore normal by Bertini's theorem. We have an ℓ-splitting

$$\mathrm{gr}_C^1 \mathscr{O} = (4P^\sharp) \mathbin{\tilde{\oplus}} \mathscr{O}_C(-F').$$

By the construction of F', we see that $(F', P) = \{v = 0\}/\boldsymbol{\mu}_m$, where $v = y_1^3 - y_2^2 + \lambda_1' y_4$ for some $\lambda_1' \in \mathscr{O}_{C,P}$ such that $\lambda_1'(P) = 0$. As in Proposition 2.9, we see that $\mathscr{O}_{F'}(-C)$ is an ℓ-invertible $\mathscr{O}_{F'}$-module with an ℓ-free ℓ-basis u at P and there exists an ℓ-isomorphism

$$\mathscr{O}_C \mathbin{\tilde{\otimes}} \mathscr{O}_{F'}(-C) \simeq (4P^\sharp).$$

We similarly see

$$H^0(\mathscr{O}_{F'}(-\nu C)) \twoheadrightarrow H^0(\mathscr{O}_C \tilde{\otimes} \mathscr{O}_{F'}(-\nu C)) \quad \text{for all } \nu \geq 0.$$

We note that y_1^2u and y_2u^2 are bases of $\mathscr{O}_C \tilde{\otimes} \mathscr{O}_{F'}(-\nu C)$ at P for $\nu = 1$ and 2, respectively. Thus, for arbitrary a_1, $a_2 \in \mathbb{C}$, there exist a section $s'_0 \in H^0(\mathscr{O}_{F'}(-C))$ such that

$$s'_0 \equiv a_1y_1^2u + a_2y_2u^2 \mod (v, u^3).$$

Recall that the map $H^0(\mathscr{O}_X) \to H^0(\mathscr{O}_{F'})$ is surjective modulo $f^*\omega_Z$ [MP09, Proposition 2.1]. In our situation, sections of $f^*\omega_Z$ lifted to $\mathbb{C}^3_{y_1,y_2,y_4}$ are contained into $\wedge^2\Omega^1_X$. We claim

$$\bigwedge^2 \Omega^1_X \subset (y_1, y_2, y_4)^3 \cdot \Omega^2_{X^\sharp} \subset (y_1, y_2, y_4)^4 \cdot \omega_{F'^\sharp}. \tag{2.12.1}$$

on the index-one cover $F'^\sharp \subset X^\sharp$ of $F' \subset X$.

Note first that the local coordinates of X at P are

$$y_1y_2, \quad y_1^5, \quad y_2^5, \quad y_1^2y_4, \quad y_2^3y_4, \quad y_2y_4^2.$$

Since y_1y_2 is the only term of degree 2, and the rest are of degree ≥ 3, we see that $\wedge^2\Omega^1_X \subset (y_1, y_2, y_4)^3 \cdot \Omega^2_{X^\sharp}$, the first inclusion.

Since $\phi = \beta_1(y_1^3 - y_2^2) + \beta_2y_4$ with β_1, $\beta_2 \in \mathscr{O}_X$ such that $\beta_2(P) = 0$, we have $\Omega^2_{X^\sharp}|_{F'^\sharp} \subset (y_1, y_2, y_4) \cdot \omega_{F'^\sharp}$ because

$$\Omega := \left.\frac{\mathrm{d}\,y_2 \wedge \mathrm{d}\,y_4}{\partial\phi/\partial y_1}\right|_{F'^\sharp} = \pm\left.\frac{\mathrm{d}\,y_1 \wedge \mathrm{d}\,y_4}{\partial\phi/\partial y_2}\right|_{F'^\sharp} = \pm\left.\frac{\mathrm{d}\,y_1 \wedge \mathrm{d}\,y_2}{\partial\phi/\partial y_4}\right|_{F'^\sharp} \in \omega_{F'^\sharp},$$

which settles the second inclusion.

From (2.12.1) and $(v, u^3) \subset (y_1^3, y_2^2, y_4^3)$ we see that there exists $s' \in H^0(I_C)$ such that

$$s' \equiv a_1y_2y_4 + a_2y_2y_4^2 \mod (y_1, y_2, y_4)^4 + (y_1^3, y_2^2, y_4^3).$$

By this, we obtain non-vanishing of the coefficient of $x_2x_3^2$ in [KM92, 10.7]. Note that [KM92, 10.7.1] is satisfied because $\lambda_1(P) \neq 0$ and [KM92, 10.7.3] is satisfied because the term y_2^5 appears and $y_1^2y_2^2$ does not appear in s_2. The rest is the same as 2.11.

2.12.2. Remark. In [KM92], the explanation at the beginning of [KM92, 8.11] was not appropriate; the non-vanishing of the coefficient of $x_2x_3^2$ of [KM92, 10.7] as well as [KM92, 10.7.3] should have been verified. The last three lines of our 2.12 supplements the insufficient treatment in [KM92, 8.11].

2.13. Case 2.10.4. Then $m = 5$ and $\lambda_1(P) = \alpha(P) = 0$. Since $\lambda_1(P) = 0$, we have $\mu_1(P) \neq 0$ because u is an ℓ-basis (see (2.2.2)). Since $\alpha(P) = 0$, we have $\alpha y_2 = \lambda_2y_1^4$ for some $\lambda_2 \in \mathscr{O}_{C,P}$ as in Lemma 2.7. Thus a general section $s \in H^0(I_C)$ satisfies the following relation near P:

$$s \equiv (\text{unit}) \cdot y_1^2(u + \lambda_2y_1^2y_4^2) \mod \mathrm{F}^3(\mathscr{O}, J). \tag{2.13.1}$$

Hence s does not contain any of the terms y_1y_2, $y_1^2y_4$, $y_2y_4^2$ and contains terms y_1^5, $y_1^2y_2^2$. By the lemma below s contains also $y_2^3y_4$.

2.13.2. Lemma. *Let τ be the weight $\tau = \frac{1}{5}(4,1,2)$ and let $(H,P) \subset \mathbb{C}^3/\boldsymbol{\mu}_5(2,3,1)$ be a normal surface singularity given by $\phi(x_1,x_2,x_3)=0$, where ϕ is a $\boldsymbol{\mu}_5$-invariant that does not contain any terms of τ-weight < 2. Then (H,P) is not a rational singularity.*

Proof. According to [Elk78] we may assume that the coefficients of ϕ are general under the assumption $\phi_{\tau=1}=0$. Consider the weighted blowup with weight τ. The exceptional divisor Υ is given in $\mathbb{P}(4,1,2)$ by the equation $\phi_{\tau=2}(x_1,x_2,x_3)=0$ or, equivalently, in $\mathbb{P}(2,1,1)$ by $\phi_{\tau=2}(x_1,x_2^{1/2},x_3)=0$. Thus, $\Upsilon \in |\mathscr{O}_{\mathbb{P}(2,1,1)}(5)|$ is a general member. By Bertini's theorem Υ is smooth and the pair $(\mathbb{P}(2,1,1),\Upsilon)$ is PLT. By the subadjunction formula

$$2p_a(\Upsilon)-2 = (K_{\mathbb{P}(2,1,1)}+\Upsilon)\cdot\Upsilon - \frac{1}{2} = 2.$$

Hence, Υ is not rational. □

2.13.3. Lemma. *The equation s contains the term $y_1y_4^3$.*

Proof. Since $\alpha(P)=0$, we can write $\alpha = y_1y_2\beta$ for some $\beta \in \mathscr{O}_{C,P}$. The unique $\mathscr{O} \subset \operatorname{gr}_C^0 J$ is generated near P by

$$y_1^2u + (y_1y_2\beta)y_2y_4^2 = y_1^2u + y_1^4\beta y_4^2 = y_1^2(u+y_1\beta y_4^2) \in \mathrm{F}^3(\mathscr{O},J).$$

By Lemma 2.7 the sequence (2.5.1) splits and we have

$$\begin{array}{l} \operatorname{gr}_C^0 J \simeq (4P^\sharp)\,\tilde{\oplus}\, \mathscr{O}_C(-2F) \\ \qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad (-1+(3P^\sharp)). \end{array}$$

Let $\mathscr{K}$ be the C-laminal ideal such that $J \supset \mathscr{K} \supset \mathrm{F}^3(\mathscr{O}_C,J)$ and $\mathscr{K}\,/\,\mathrm{F}^3(\mathscr{O},J) = (4P^\sharp)$. Then $\mathscr{K}$ is locally a nested c.i. on $C\setminus\{P\}$ and (y_4,u) is a $(1,3)$-monomializable ℓ-basis of $I_C \supset \mathscr{K}$ at P (where u is given by (2.2.2)). We have

$$\begin{array}{ccccccc} 0 \to & (-1+2P^\sharp) & \longrightarrow & \operatorname{gr}_C^0\mathscr{K} & \longrightarrow & (4P^\sharp) & \to 0 \\ & \| & & & & & \\ & \mathscr{O}_C(-3F) & & & & & \end{array}$$

Since $H^1(\mathscr{O}_C(-3F)\,\tilde{\otimes}\,\omega) \neq 0$, as in the proof of Proposition 2.8 the sequence does not split. So, locally near P, the sheaf $\operatorname{gr}_C^0\mathscr{K}$ has a section $y_1^2u+\gamma y_1y_4^3$ with $\gamma(P)\neq 0$. □

Thus, by the two lemmas 2.13.2 and 2.13.3 above, s does not contain any of the terms y_1y_2, $y_1^2y_4$, $y_2y_4^2$ and contains terms y_1^5, $y_1^2y_2^2$, $y_2^3y_4$, $y_1y_4^3$. Therefore, [KM92, 10.8] can be applied to (H,P). It is easy to see that the whole configuration contracts to a curve. We get 1.3.1. This completes the proof of Theorem 1.3.

3. Case (IIB)

3.1. Setup. Let (X, P) be the germ of a three-dimensional terminal singularity and let $C \subset (X, C)$ be a smooth curve. Recall that the triple (X, C, P) is said to be of type (IIB) if (X, P) is a terminal singularity of type cAx/4 and there are analytic isomorphisms

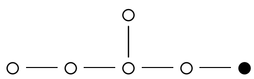
$$(X, P) \simeq \{y_1^2 - y_2^3 + \alpha = 0\}/\boldsymbol{\mu}_4(3, 2, 1, 1) \subset \mathbb{C}^4_{y_1,\dots,y_4}/\boldsymbol{\mu}_4(3, 2, 1, 1),$$
$$C \simeq \{y_1^2 - y_2^3 = y_3 = y_4 = 0\}/\boldsymbol{\mu}_4(3, 2, 1, 1),$$

where $\alpha = \alpha(y_1, \dots, y_4) \in (y_3,\, y_4)$ is a semi-invariant with $\operatorname{wt} \alpha \equiv 2 \mod 4$ and $\alpha_2(0, 0, y_3, y_4) \neq 0$ (see [Mor88, A.3]).

3.1.1. Definition. We say that (X, P) is a *simple* (resp. *double*) cAx/4-point if $\operatorname{rk} \alpha_2(0, 0, y_3, y_4) = 2$ (resp. $\operatorname{rk} \alpha_2(0, 0, y_3, y_4) = 1$).

3.1.2. Let (X, C) be an extremal curve germ and let $f\colon (X, C) \to (Z, o)$ be the corresponding contraction. In this section we assume that C is irreducible and has a point P of type (IIB). According to [KM92, Theorem 4.5] the germ (X, C) is not flipping. Recall that (X, C) is locally primitive at P [Mor88, 4.2]. Moreover, P is the only singular point on [Mor88, Theorem 6.7], [MP08, Theorem 8.6, Lemma 7.1.2]. Thus the group $\operatorname{Cl}(Z, o)$ has no torsion. Therefore, f is either a divisorial contraction to a cDV point or a conic bundle over a smooth base.

3.2. According to [KM92, Theorem 2.2] and [MP09] a general member $F \in |-K_X|$ contains C, has only Du Val singularities, and the graph $\Delta(F, C)$ has the form



where all the vertices correspond to (-2)-curves and $\bullet$ corresponds to C. Under the identifications of 3.1, a general member $F \in |-K_X|$ near P is given by $\lambda y_3 + \mu y_4 = 0$ for some λ, $\mu \in \mathscr{O}_X$ such that $\lambda(0)$, $\mu(0)$ are general in $\mathbb{C}^*$ [KM92, 2.11], [MP09, §4].

3.3. Let H be a general member of $|\mathscr{O}_X|_C$, let $T := f(H)$, and let $\Gamma := H \cap F$.

3.3.1. If f is divisorial, we put $F_Z := f(F)$ and $\Gamma_Z := f(\Gamma)$. Then $F_Z \in |-K_Z|$, T is a general hyperplane section of (Z, o) and Γ_Z is a general hyperplane section of F_Z.

3.3.2. If f is a $\mathbb{Q}$-conic bundle, we consider the Stein factorization

$$f_F : (F, C) \xrightarrow{f_1} (F_Z, o_Z) \xrightarrow{f_2} (Z, o).$$

Here we put $\Gamma_Z := f_1(\Gamma)$.

In both cases F_Z is a Du Val singularity of type E_6 by 3.2.

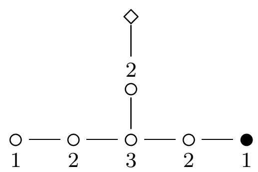
3.4. Lemma.

(i) *H is normal, has only rational singularities, and smooth outside of P;*
(ii) *$\Gamma = C + \Gamma_1$ (as a scheme), where Γ_1 is a reduced irreducible curve;*
(iii) *if f is birational, then $T = f(H)$ is Du Val singularity of type E_6, D_5, D_4, A_4,...,A_1 (or smooth).*

Proof. Consider two cases:

3.4.1. Case: f is divisorial. Since the point (Z, o) is terminal of index 1, the germ (T, o) is a Du Val singularity. Since Γ_Z is a general hyperplane section of F_Z, we that the graph $\Delta(F, \Gamma)$ has the form

(3.4.2)



where, as usual, $\diamond$ corresponds to the proper transform of Γ_Z and numbers attached to vertices are coefficients of corresponding exceptional curves in the pull-back of Γ_Z. By Bertini's theorem H is smooth outside of C. Since the coefficient of C equals to 1, $F \cap H = C + \Gamma$ (as a scheme), so H is smooth outside of P. In particular, H is normal. Since $f_H : H \to T$ is a birational contraction and (T, o) is a Du Val singularity, the singularities of H are rational.

3.4.3. Case: f is a $\mathbb{Q}$-conic bundle. We may assume that, in some coordinate system, the germ (F_Z, o_Z) is given by $x^2 + y^3 + z^4 = 0$. Then by [Cat87] up to coordinate change the double cover $(F_Z, o_Z) \longrightarrow (Z, o)$ is just the projection to the (y, z)-plane. Hence we may assume that Γ_Z is given by $z = 0$. As in the case 3.4.1 we see that the graph $\Delta(F, \Gamma)$ has the form (3.4.2). Therefore, H is smooth outside of P. The restriction $f_H : H \to T$ is a rational curve fibration. Hence H has only rational singularities.

(iii) follows by the fact that there is a hyperplane section F_Z of (Z, o) which is Du Val of type E_6 (see e.g. [Arn72]).

$\square$

We need a more detailed description of (H, C) near P.

3.4.4. Lemma. *In the notation of* 3.1 *the surface $H \subset X$ is locally near P given by the equation $y_3 v_3 + y_4 v_4 = 0$, where v_3, $v_4 \in \mathscr{O}_{P^\sharp, X^\sharp}$ are semi-invariants with $\operatorname{wt} v_i \equiv 3$ and at least one of v_3 or v_4 contains a linear term in y_1.*

Proof. Since H is normal and $\operatorname{gr}^1_C \mathscr{O} \simeq \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$, we have $\mathscr{O}_C(-H) = \mathscr{O} \subset \operatorname{gr}^1_C \mathscr{O}$, i.e. the local equation of H must be a generator of $\mathscr{O} \subset \operatorname{gr}^1_C \mathscr{O}$. $\square$

3.5. Let σ be the weight $\frac{1}{4}(3,2,1,1)$. By Lemma 3.4.4 the surface germ (H,P) can be given in $\mathbb{C}^4/\boldsymbol{\mu}_4(3,2,1,1)$ by two equations:

$$
\begin{cases} y_1^2 - y_2^3 + \eta(y_3,y_4) + \phi(y_1,y_2,y_3,y_4) = 0, \\ y_1 l(y_3,y_4) + y_2 q(y_3,y_4) + \xi(y_3,y_4) + \psi(y_1,y_2,y_3,y_4) = 0, \end{cases} \tag{3.5.1}
$$

where η, l, q and ξ are homogeneous polynomials of degree 2, 1, 2 and 4, respectively, $\eta \neq 0$, $l \neq 0$, ϕ, $\psi \in (y_3,\ y_4)$, $\sigma\text{-}\operatorname{ord}\phi \geq 3/2$, $\sigma\text{-}\operatorname{ord}\psi \geq 2$. Moreover, $\operatorname{rk}\eta = 2$ (resp. $\operatorname{rk}\eta = 1$) if (X,P) is a simple (resp. double) cAx/4-point.

3.5.2. Consider the weighted blowup

$$
g : (W \supset \tilde{X} \supset \tilde{H}) \longrightarrow (\mathbb{C}^4/\boldsymbol{\mu}_4(3,2,1,1) \supset X \supset H)
$$

with weight σ. Let E be the g-exceptional divisor, let $\Xi := E \cap \tilde{H}$ be the exceptional divisor of $g_H := g|_{\tilde{H}}$, and let $\tilde{C}$ be the proper transform of C. Denote

$$
\Xi_0 := \{y_3 = y_4 = 0\} \subset E.
$$

If $\tilde{H}$ is normal, let $g_1 : \hat{H} \to \tilde{H}$ be the minimal resolution. Thus, in this case, we have the following morphisms

$$
h : \hat{H} \xrightarrow{g_1} \tilde{H} \xrightarrow{g_H} H \xrightarrow{f_H} T.
$$

3.5.3. Lemma.

(i) $E \simeq \mathbb{P}(3,2,1,1)$ *and* Ξ *is given in this* $\mathbb{P}(3,2,1,1)$ *by*

$$
\eta(y_3,y_4) = y_1 l(y_3,y_4) + y_2 q(y_3,y_4) + \xi(y_3,y_4) = 0;
$$

(ii) $\tilde{C}$ *of* C *meets* E *at* $Q := (1:1:0:0) \in \Xi_0$;

(iii) Ξ_0 *is a component of* Ξ *and* $(\Xi_0 \cdot \Xi)_{\tilde{H}} = -2/3$;

(iv) *If* $\tilde{H}$ *is normal, then* $K_{\tilde{H}} = g^* K_H - \frac{3}{4}\Xi$.

Proof. Statements (i) and (ii) are obvious, (iii) follows from

$$
(\Xi_0 \cdot \Xi)_{\tilde{H}} = (\Xi_0 \cdot E)_W = (\Xi_0 \cdot \mathscr{O}_E(E))_E = (\Xi_0 \cdot \mathscr{O}_E(-4))_E = -\frac{2}{3},
$$

and (iv) follows from $K_W = g^* K_{\mathbb{C}^4/\boldsymbol{\mu}_4} + \frac{3}{4}E$. □

3.6. Case of simple cAx/4**-point.** After a coordinate change we may assume that $\eta = y_3 y_4$. We also may assume that the term y_3 appears in $l(y_3,y_4)$ with coefficient 1, that is, $l(y_3,y_4) = y_3 + c y_4$, $c \in \mathbb{C}$. Thus the equations (3.5.1) for (H,P) have the form:

$$
\begin{cases} y_1^2 - y_2^3 + y_3 y_4 + \phi = 0, \\ y_1(y_3 + c y_4) + y_2 q(y_3,y_4) + \xi(y_3,y_4) + \psi = 0. \end{cases} \tag{3.6.1}
$$

It is easy to see that in this case $\tilde{X}$ has only isolated (terminal) singularities. Indeed, $\tilde{X} \cap E$ is given by $y_3 y_4 = 0$ in $E \simeq \mathbb{P}(3,2,1,1)$. Hence, $\operatorname{Sing}(\tilde{X}) \subset \Xi_0 \cup \operatorname{Sing}(E)$. There are the following subcases.

3.6.2. Subcase: (X, P) **is simple** $\mathrm{cAx}/4$**-point and** $c \neq 0$**.** We shall show that only the case 1.4.1 occurs. We may assume that in (3.6.1) $l(y_3, y_4) = y_3 + y_4$. In this case, $\Xi = 2\Xi_0 + \Xi' + \Xi''$, where Ξ' and Ξ'' are given in $E \simeq \mathbb{P}(3, 2, 1, 1)$ as follows:

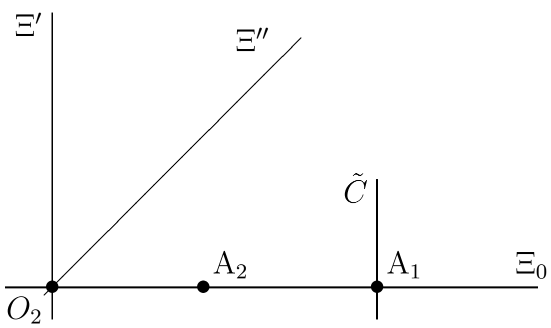
$$\begin{aligned} \Xi' &:= \quad \{y_3 = y_1 + y_2 q(0, y_4)/y_4 + \xi(0, y_4)/y_4 = 0\}, \\ \Xi'' &:= \quad \{y_4 = y_1 + y_2 q(y_3, 0)/y_3 + \xi(y_3, 0)/y_3 = 0\}. \end{aligned}$$

All the components of Ξ pass through $(0 : 1 : 0 : 0)$ and do not meet each other elsewhere.

3.6.2.1. Claim. *The surface* $\tilde{H}$ *is normal and has the following singularities (in natural weighted coordinates on* $E \simeq \mathbb{P}(3, 2, 1, 1)$*):*

- $O_1 := (1 : 0 : 0 : 0)$ *which is of type* A_2*,*
- $Q := \Xi_0 \cap \tilde{C} = (1 : 1 : 0 : 0)$ *which is of type* A_1*,*
- $O_2 := \Xi_0 \cap \Xi' \cap \Xi'' = (0 : 1 : 0 : 0)$ *which is a log terminal point of index 2 (a cyclic quotient singularity of type* $\frac{1}{4k}(1, 2k - 1)$*).*

Pairs $(\tilde{H}, \Xi_0 + \Xi' + \tilde{C})$ *and* $(\tilde{H}, \Xi_0 + \Xi'' + \tilde{C})$ *are log canonical (LC). Moreover, they are purely log terminal (PLT) at all points of* $\Xi_0 \setminus \{O_2, Q\}$*. Thus the surface* $\tilde{H}$ *looks as follows:*



Proof. Since $\Xi = \tilde{H} \cap E$ is reduced along Ξ' and Ξ'', the singular locus of $\tilde{H}$ is contained in $\Xi_0 = \{y_3 = y_4 = 0\}$.

Consider the chart $U_1 = \{y_1 \neq 0\} \subset W$, $U_1 \simeq \mathbb{C}^4/\boldsymbol{\mu}_3(2, 2, 1, 1)$. The equations of $\tilde{H}$ have the form

$$\begin{cases} y_1 - y_1 y_2^3 + y_3 y_4 + y_1 \phi_{3/2}(1, y_2, y_3, y_4) + y_1^2(\cdots) = 0, \\ y_3 + y_4 + y_2 q(y_3, y_4) + \xi(y_3, y_4) + y_1 \psi_2(1, y_2, y_3, y_4) + y_1^2(\cdots) = 0. \end{cases}$$

and $\tilde{C}$ is cut out on $\tilde{H}$ by $y_3 = y_4 = 0$. Using the condition $y_1 = y_3 = y_4 = 0$ one can obtain that the surface $\tilde{H} \cap U_1$ has on the exceptional divisor $\{y_1 = 0\}$ two singular points: $Q = \{y_1 = y_3 = y_4 = 1 - y_2^3 = 0\}$ and the origin O_1. It is easy to see that $(\tilde{H}, Q)$ is a Du Val singularity of type A_1 and $(\tilde{H}, O_1)$ is a Du Val singularity of type A_2. Since Ξ_0 and $\tilde{C}$ are smooth curves meeting each other transversely, the pair $K_{\tilde{H}} + \Xi_0 + \tilde{C}$ is LC at Q.

Consider the chart $U_2 = \{y_2 \neq 0\} \subset W$, $U_2 \simeq \mathbb{C}^4/\boldsymbol{\mu}_2(1,0,1,1)$. The equations of $\tilde{H}$ have the form

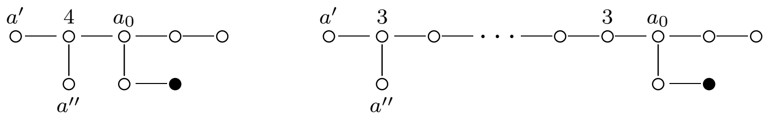
$$\begin{cases} y_1^2 y_2 - y_2 + y_3 y_4 + y_2\phi_{3/2}(y_1, 1, y_3, y_4) + y_2^2(\cdots) = 0, \\ y_1(y_3+y_4) + q(y_3,y_4) + \xi(y_3,y_4) + y_2\psi_2(y_1,1,y_3,y_4) + y_2^2(\cdots) = 0. \end{cases}$$

Then we get only one new singular point: the origin O_2 where the singularity of $\tilde{H}$ is analytically isomorphic to a singularity in $\mathbb{C}^3_{y_1,y_3,y_4}/\boldsymbol{\mu}_2(1,1,1)$ given by

$$\{y_1(y_3+y_4) + q(y_3,y_4) + (\text{terms of degree} \geq 3) = 0\}. \tag{3.6.3}$$

Hence, $(\tilde{H}, O_2)$ is a log terminal singularity of index 2. $\square$

Therefore, for the graph $\Delta(\hat{H}, \Gamma + \hat{C})$ we have only the following two possibilities:



where the vertex marked by a_0 (resp. a', a'') corresponds to Ξ_0 (resp. Ξ', Ξ'') and $\bullet$ corresponds to $\hat{C}$.

Using Lemma 3.5.3, (iii) one can easily obtain that $a_0 = 2$. Similarly,

$$(\Xi' \cdot \Xi)_{\tilde{H}} = (\Xi'' \cdot \Xi)_{\tilde{H}} = -2.$$

This gives us $a' = a'' = 3$. However the second of the above configurations is not contractible. We get the case 1.4.1.

3.6.4. Corollary. $q(0, y_4) \neq 0$.

Proof. Assume that $q(0,y_4) = 0$. Take H so that, in (3.5.1), functions η, ϕ, l, q, ξ, and ψ are sufficiently general under this assumption. Let X' be a general one-parameter deformation family of H. According to [KM92, Prop. 11.4] there is a contraction $f' : X' \to Z'$, so (X', C) is an extremal curve germ. Moreover, (X', C') is of type IIB. By 3.6.2 we get a contradiction (otherwise (3.6.3) is not a point of type $\frac{1}{4}(1,1)$). $\square$

3.6.5. Subcase: (X, P) is simple cAx/4-point and $c = 0$. We shall show that only the case 1.4.2 occurs. Equations (3.6.1) have the form

$$\begin{cases} y_1^2 - y_2^3 + y_3 y_4 + \phi = 0, \\ y_1 y_3 + y_2 q(y_3, y_4) + \xi(y_3,y_4) + \psi = 0. \end{cases}$$

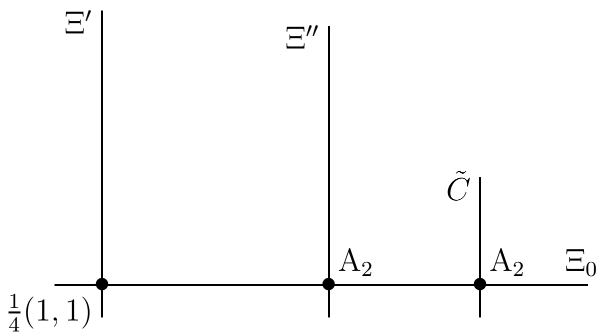
In this case, $\Xi = 3\Xi_0 + \Xi' + \Xi''$, where Ξ' and Ξ'' are given in $E \simeq \mathbb{P}(3,2,1,1)$ as follows:

$$\Xi' = \{y_4 = y_1 + y_2 q(y_3, 0)/y_3 + \xi(y_3,0)/y_3 = 0\},$$
$$\Xi'' = \{y_3 = y_2 q(0, y_4)/y_4^2 + \xi(0,y_4)/y_4^2 = 0\}.$$

3.6.5.1. Claim. *The surface $\tilde{H}$ is normal and has the following singularities (in natural weighted coordinates on $E \simeq \mathbb{P}(3,2,1,1)$):*

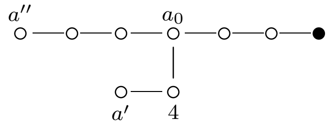
- $O_1 := \Xi_0 \cap \Xi'' = (1:0:0:0)$ *which is of type* A_2,
- $Q := \Xi_0 \cap \tilde{C} = (1:1:0:0)$ *which is of type* A_2,
- $O_2 := \Xi_0 \cap \Xi' = (0:1:0:0)$ *which is of type* $\frac{1}{4}(1,1)$.

The pair $(\tilde{H}, \Xi_0 + \Xi' + \Xi'' + \tilde{C})$ *is LC. Thus* $\tilde{H}$ *looks as follows*:



The proof is similar to the proof of Claim 3.6.2.1, so we omit it.

By the above claim $\Delta(H, C)$ has the form



Since

$$(\Xi' \cdot \Xi)_{\tilde{H}} = -2, \quad (\Xi'' \cdot \Xi)_{\tilde{H}} = -\frac{4}{3}.$$

(cf. Lemma 3.5.3, (iii)), we have $a_0 = 2$ and $a' = a'' = 3$. Thus we get the case 1.4.2.

3.7. Case of double cAx/4**-point.** We may assume that $\eta = y_3^2$. By Corollary 3.6.4 $q(0, y_4) \neq 0$, so we also may assume that $q(0, y_4) = y_4^2$. Thus the equations (3.5.1) for (H, P) have the form:

$$\begin{cases} y_1^2 - y_2^3 + y_3^2 + \phi = 0, \\ y_1 l(y_3,\, y_4) + y_2 q(y_3, y_4) + \xi(y_3, y_4) + \psi = 0. \end{cases}$$

where ϕ does not contain any terms of degree ≤ 2. This case is more complicated because $\tilde{X}$ has non-isolated singularities:

3.7.1. Remark. $\mathrm{Sing}(\tilde{X})$ has exactly one one-dimensional irreducible component

$$\Lambda := \{y_3 = y_1^2 - y_2^3 + \phi_{\sigma=3/2}(y_1, y_2, 0, y_4) = 0\} \subset E \simeq \mathbb{P}(3,2,1,1).$$

There are the following subcases.

3.7.2. Subcase: (X, P) **is double** cAx/4**-point and** $l(0,\, y_4) \neq 0$. We shall show that only the case 1.4.3 occurs. After a coordinate change, we may assume that $l(y_3,\, y_4) = y_4$, so the equations (3.5.1) for

(H, P) have the form:

$$\begin{cases} y_1^2 - y_2^3 + y_3^2 + \phi = 0, \\ y_1 y_4 + y_2 q(y_3, y_4) + \xi(y_3, y_4) + \psi = 0. \end{cases} \tag{3.7.3}$$

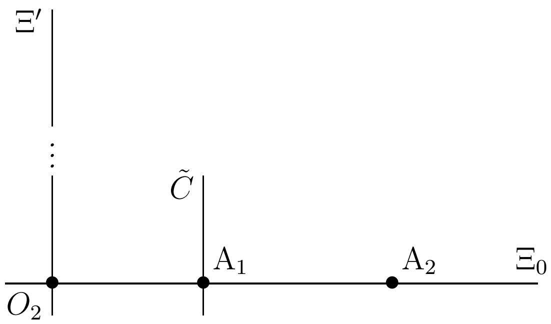
In this case, $\Xi = 2\Xi_0 + 2\Xi'$, where

$$\Xi' = \{y_3 = y_1 + y_2 q(0, y_4)/y_4 + \xi(0, y_4)/y_4 = 0\} \subset E \simeq \mathbb{P}(3, 2, 1, 1).$$

3.7.3.1. Claim. *The surface $\tilde{H}$ is normal and has the following singularities on Ξ_0 (in natural weighted coordinates on $E \simeq \mathbb{P}(3, 2, 1, 1)$):*

- $O_1 := (1 : 0 : 0 : 0)$ *which is of type* A_2,
- $Q := \Xi_0 \cap \tilde{C} = (1 : 1 : 0 : 0)$ *which is of type* A_1,
- $O_2 := \Xi_0 \cap \Xi' = (0 : 1 : 0 : 0)$ *which is a log terminal point of index* 2.

The pair $(\tilde{H}, \Xi_0 + \Xi' + \tilde{C})$ is LC along Ξ_0. Moreover, it is PLT at all points of $\Xi_0 \setminus \{O_2, Q\}$. Thus $\tilde{H}$ looks as follows:

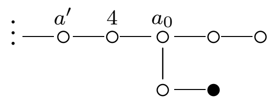


where there are more singular points sitting on $\Xi' \setminus \{O_2\}$ which must be Du Val.

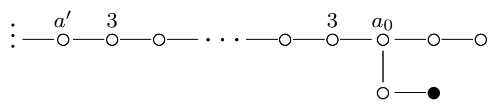
3.7.3.2. Remark. For general choice of ξ and ϕ the surface $\tilde{H}$ has exactly three singular points on $\Xi' \setminus \{O_2\}$ and these points are of type A_1.

Hence the dual graph $\Delta(H, C)$ has one of the following forms:

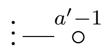
a)



b)



where $\vdots$ corresponds to some Du Val singularities sitting on Ξ'. Since the whole configuration is contractible to either a Du Val point or a curve, we have $a_0 = 2$ and the case b) does not occur. In the case a), contracting black vertices successively we get



Hence $a' = 2$ or 3.

3.7.3.3. Let (S, o) be a normal surface singularity and let $\mu : \hat{S} \to S$ be its resolution. Recall that the *codiscrepancy divisor* is a unique $\mathbb{Q}$-divisor $\Theta = \sum \theta_i \Theta_i$ on $\hat{S}$ with support in the exceptional locus such that $\mu^* K_S = K_{\hat{S}} + \Theta$. If μ is the minimal resolution, then Θ must be effective. The coefficient θ_i is called the *codiscrepancy* of Θ_i. We denote it by $\operatorname{cdisc}(\Theta_i)$. If (S, o) is a rational singularity, then $\theta_i = \operatorname{cdisc}(\Theta_i)$ can be found from the following system of linear equations:

$$\sum_i \theta_i \Theta_i \cdot \Theta_j = -K_{\hat{S}} \cdot \Theta_j = 2 + \Theta_j^2.$$

Let $a_i := -\Theta_i^2$. Then the system can be rewritten as follows:

$$a_j \theta_j = -\Theta_j^2 - 2 + \sum{}' \theta_i$$

where $\sum'$ runs through all exceptional curves Θ_i meeting Θ_j.

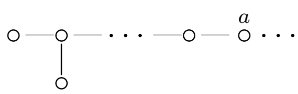
3.7.3.4. Corollary. *Let Δ be the dual graph of a resolution of a rational singularity and let Δ' be its subgraph consisting of one vertex of weight $a \geq 2$ and $n-1$ vertices of weight 2. Assume that the remaining part $\Delta \setminus \Delta'$ is attached to $\overset{a}{\circ}$.*

(i) *If Δ' has the form*

$$\circ - \cdots - \circ - \overset{a}{\circ} \cdots$$

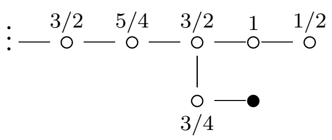
then the codiscrepancies of the corresponding to Δ' exceptional components, indexed from the left to right, are computed by $\alpha_k = k\alpha_1$, $k \leq n$.

(ii) *If Δ' has the form*



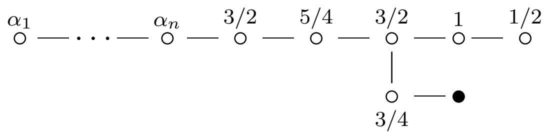
then the codiscrepancies of the corresponding to Δ' exceptional components are computed by $\alpha_1 = \alpha_2 = 2\alpha_3$ and $\alpha_k = \alpha_3$ for $3 \leq k \leq n$.

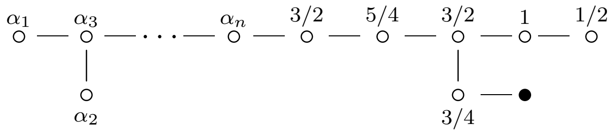
3.7.3.5. By Lemma 3.5.3, (iv) we have $\operatorname{cdisc}(\Xi_0) = \operatorname{cdisc}(\Xi') = 3/2$. Using 3.7.3.3 we compute the codiscrepancies of exceptional divisors over $\widetilde{H}$:



3.7.3.6. If $a' = 2$, then the configuration $\vdots - \overset{a'-1}{\circ}$ is contracted either to a smooth point or to a curve. Therefore we have one of the following

possibilities:

a1) 

a2) 

$n \geq 2$

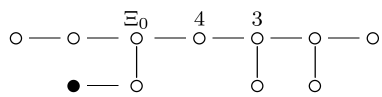
Then we get a contradiction by Corollary 3.7.3.4.

3.7.3.7. Thus, $a' = 3$. Then f is divisorial and the configuration $\vdots\!\!-\!\!\overset{a'-1}{\circ}$ is exactly the dual graph of the minimal resolution of (T, o) which is a Du Val graph of type E_6, D_5, D_4, A_4, A_3, A_2 or A_1. If the graph $\Delta(H, C)$ has the form a1), then, as above, $3/2 = \alpha_{n+1} = (n+1)\alpha_1$, $3 \cdot 3/2 = 1 + \alpha_n + 5/4$. This gives us $n\alpha_1 = 9/4$, $\alpha_1 = 3/2 - 9/4 < 0$, a contradiction. Similarly, in the case a2) with $n \geq 3$ we obtain $\alpha_n = 3/2$, $3 \cdot 3/2 = 1 + \alpha_n + 5/4$, a contradiction.

If there are three connected components of the exceptional divisor attached to Ξ', then for corresponding codiscrepancies α_n, β_m, γ_l we have $3 \cdot 3/2 = 1 + \alpha_n + \beta_m + \gamma_l + 5/4$, $\alpha_n + \beta_m + \gamma_l = 9/4$. On the other hand, $2\alpha_n \geq 3/2$, $2\beta_m \geq 3/2$, $2\gamma_l \geq 3/2$. Hence the equalities $\alpha_n = \beta_m = \gamma_l = 3/4$ hold and we the get case 1.4.3.

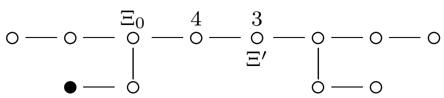
In the remaining cases, by direct computations we obtain that the exceptional divisors have codiscrepancies whose denominators divide 4 only in cases 3.7.3.8 or 3.7.3.9 below.

3.7.3.8. (T, o) is Du Val of type D_5 and $\Delta(H, C)$ has the form



here $\tilde{H}$ has two singular points on $\Xi' \setminus \Xi_0$ and these points are of types A_1 and A_3.

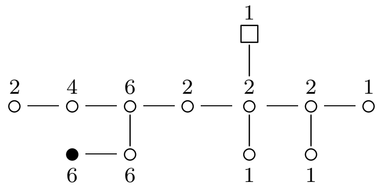
3.7.3.9. (T, o) is Du Val of type E_6 and $\Delta(H, C)$ has the form



here $\tilde{H}$ has exactly one singular points on $\Xi' \setminus \Xi_0$ and this point is of type A_5.

3.7.4. Now we show that in cases 3.7.3.8 and 3.7.3.9 the chosen element $H \in |\mathscr{O}_X|_C$ is not general. Consider the case 3.7.3.8. Case 3.7.3.9 can

be treated similarly. Take a divisor D on $\hat{H}$ whose coefficients are as follows:



where $\square$ corresponds to an arbitrary smooth analytic curve $\hat{G}$ meeting Ξ' transversely so that $\operatorname{Supp} D$ is a simple normal crossing divisor. It is easy to verify that D is numerically trivial, so $D = h^*G_Z$, where G_Z is a Cartier divisor on T. Since $R^1 f_* \mathscr{O}_X = 0$, by the exact sequence

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_X(H') \longrightarrow \mathscr{O}_H(H') \longrightarrow 0$$

we get surjectivity of the map $H^0(X, \mathscr{O}_X(H')) \to H^0(H, \mathscr{O}_H(H'))$. Thus there is a member $H' \in |\mathscr{O}_X|_C$ such that $H'|_H = f_H^* G_Z$.

The proper transform $\tilde{H}'$ of H' by g satisfies $\tilde{H}' = g^*H' - E|_{\tilde{X}}$. Since $\Xi = E \cap \tilde{H}$ and $\Xi = 2\Xi_0 + 2\Xi'$, we have $\tilde{H}'|_{\tilde{H}} = 4\Xi_0 + g_1(\hat{G})$. In particular, Ξ' is not a component of $\tilde{H}'|_{\tilde{H}}$. Note that $|g_1(\hat{G})|$ is a base point free linear system on $\tilde{H}$ (because $H^1(\mathscr{O}_{\tilde{H}}) = 0$). Thus we can take H' so that $\tilde{H}'$ does not pass through points in $\tilde{H} \cap \Lambda \setminus \Xi_0$. Now let H_ϵ be a general member of the pencil generated by H and H'. Note that $\Lambda \cap \Xi_0 = \{Q\}$ and Λ meets $\tilde{H}$ and $\tilde{H}_\epsilon$ transversely at Q. By Bertini's theorem the proper transform $\tilde{H}_\epsilon$ of H_ϵ on $\tilde{X}$ meets Λ transversely also along Ξ'. Since $(\tilde{H}_\epsilon \cdot \Lambda)_{\tilde{X}} = (\mathscr{O}(4) \cdot \Lambda)_{\mathbb{P}(3,2,1,1)} = 4$, the intersection $\tilde{H}_\epsilon \cap \Lambda$ consists of four distinct points. Therefore, $\tilde{H}_\epsilon$ has three Du Val points on $\tilde{H}_\epsilon \cap \Lambda \setminus \Xi_0$. This shows that for H_ϵ the situation of 1.4.3 holds, so the chosen H is not general in the case 3.7.3.8.

3.7.5. Subcase: (X, P) **is double** cAx/4**-point and** $l(0, y_4) = 0$**.** We shall show that only the case 1.4.4 occurs. We may assume that $l(y_3, y_4) = y_3$, so the equations (3.5.1) for (H, P) have the form:

$$\begin{cases} y_1^2 - y_2^3 + y_3^2 + \phi = 0, \\ y_1 y_3 + y_2 q(y_3, y_4) + \xi(y_3, y_4) + \psi = 0. \end{cases} \tag{3.7.6}$$

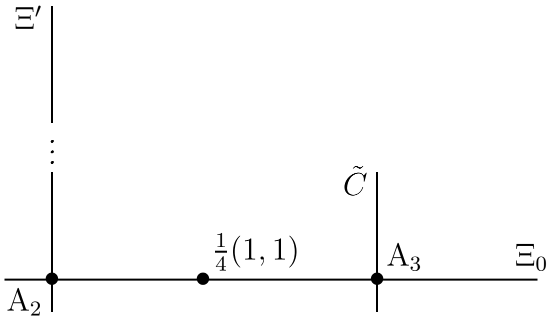
In this case, $\Xi = 4\Xi_0 + 2\Xi'$, where

$$\Xi' = \{y_3 = y_2 q(0, y_4)/y_4^2 + \xi(0, y_4)/y_4^2 = 0\} \subset E \simeq \mathbb{P}(3, 2, 1, 1).$$

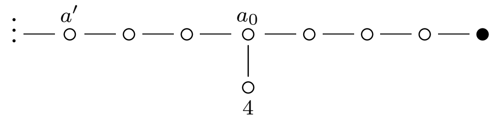
3.7.6.1. Claim. *The surface $\tilde{H}$ is normal and has the following singularities on Ξ_0 (in natural weighted coordinates on $E \simeq \mathbb{P}(3, 2, 1, 1)$):*

- $O_1 := \Xi_0 \cap \Xi' = (1 : 0 : 0 : 0)$ *which is of type* A_2*,*
- $Q := \Xi_0 \cap \tilde{C} = (1 : 1 : 0 : 0)$ *which is of type* A_3*,*
- $O_2 := (0 : 1 : 0 : 0)$ *which is a cyclic quotient singularity of type* $\frac{1}{4}(1, 1)$*.*

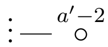
The pair $(\tilde{H}, \Xi_0 + \Xi' + \tilde{C})$ is LC along Ξ_0. Moreover, it is PLT at all points of $\Xi_0 \setminus \{O_1, Q\}$. Thus $\tilde{H}$ looks as follows:



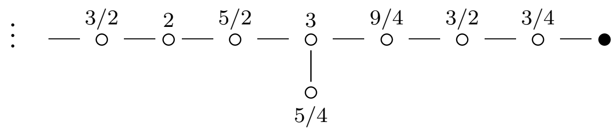
Hence the dual graph $\Delta(H, C)$ has the following form:



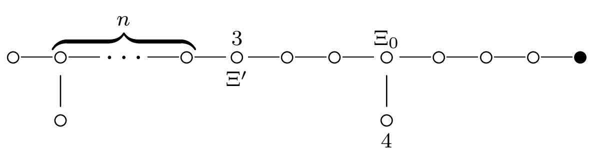
where $\vdots$ corresponds to some Du Val singularities sitting on Ξ'. Since the whole configuration is contractible to either a Du Val point or a curve, we have $a_0 = 2$. Contracting black vertices successively on some step we get



Recall that $\vdots$ is not empty. Hence $a' = 3$ or 4. By Lemma 3.5.3, (iv) we have $\operatorname{cdisc}(\Xi_0) = 3$, $\operatorname{cdisc}(\Xi') = 3/2$. Using 3.7.3.3 we compute the codiscrepancies of exceptional divisors over $\tilde{H}$:

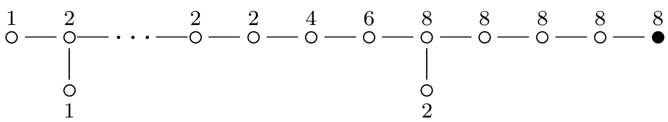


If $a' = 4$, we get a contradiction as in 3.7.3.7. If $a' = 3$, then the whole configuration contracts to a curve, i.e., f is a $\mathbb{Q}$-conic bundle. As in 3.7.3.6 we infer that the graph $\Delta(H, C)$ has the following form



where $n \geq 0$.

We show that $n = 0$, that is, the case 1.4.4 holds. As in 3.7.4 take a divisor D on $\hat{H}$ whose coefficients are as follows



Then $D = h^*o$ is a scheme fiber of $h : \hat{H} \to T$. There is a member $H' \in |\mathscr{O}_X|_C$ such that $H'|_H = g_{H*}g_{1*}D = f_H^*o$. Since $\Xi = 4\Xi_0 + 2\Xi'$, we have $\tilde{H}'|_{\tilde{H}} = g_{1*}D - \Xi = 4\Xi_0$. In particular, the curve Ξ' is not a component of $\tilde{H}'|_{\tilde{H}}$. Hence the base locus of the pencil generated by $\tilde{H}$ and $\tilde{H}'$ coincides with Ξ_0. As in 3.7.4 a general member $\tilde{H}_\epsilon$ of this pencil meets the curve Λ transversely outside of Ξ_0. Note that $\Lambda \cap \Xi_0 = \{Q\}$ and the local intersection number of Λ and $\tilde{H}_\epsilon$ at Q equals to 2. By Bertini's theorem the proper transform $\tilde{H}_\epsilon$ of H_ϵ on $\tilde{X}$ meets Λ transversely along Ξ'. Since $(\tilde{H}_\epsilon \cdot \Lambda)_{\tilde{X}} = (\mathscr{O}(4) \cdot \Lambda)_{\mathbb{P}(3,2,1,1)} = 4$, the intersection $\tilde{H}_\epsilon \cap \Lambda$ consists of three distinct points. Therefore, $\tilde{H}_\epsilon$ has two Du Val points on $\tilde{H}_\epsilon \cap \Lambda \setminus \Xi_0$. This shows that for H_ϵ the situation of 1.4.4 holds, so the chosen H is not general if $n > 0$.

3.7.6.2. Example. Let H be given by the equations

$$\begin{cases} y_1^2 - y_2^3 + y_3^2 = 0, \\ y_1y_3 + y_2y_4^2 + y_4^4 = 0. \end{cases}$$

Then a one-parameter deformation of H is a $\mathbb{Q}$-conic bundle as in 1.4.4.

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Shigefumi Mori: RIMS, Kyoto University, Oiwake-cho, Kitashirakawa, Sakyo-ku, Kyoto 606-8502, Japan
E-mail address: mori@kurims.kyoto-u.ac.jp

Yuri Prokhorov: Department of Algebra, Faculty of Mathematics, Moscow State University, Moscow 117234, Russia
Laboratory of Algebraic Geometry, SU-HSE, 7 Vavilova Str., Moscow 117312, Russia
E-mail address: prokhoro@gmail.com